5 consider $\sum_{n=0}^{\infty} 3^{-n} x^{4n}$

$$a_n = \begin{cases} 3^{-(n/4)} & \text{if } 4 \mid n \\ 0 & \text{if } 4 \nmid n \end{cases}$$

here $\frac{a_3}{a_4} = \frac{0}{3^{-1}} = 0$

$\frac{a_4}{a_5} = \frac{3^{-1}}{0}$ not well-defined

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ again: $R = 1$
(check for yourself!)

$\underline{x=1}$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\boxed{\text{converges}}$
$\left(= \pi^2/6\right)$

$\underline{x=-1}$ $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ $\boxed{\text{converges}}$

⚠ ratio test does not always work

③ $\sum_{n=1}^{\infty} \frac{1}{n} x^n$

check: $R = 1$

$$\left(\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{1/n+1}{1/n} \right| \right.$$

$$= \lim \left| \frac{n}{n+1} \right| = \lim 1 - \frac{1}{n+1} = 1$$

$$\Rightarrow \beta = 1 \Rightarrow R = 1)$$

converges for $|x| < 1$

<u>$x = 1$</u>: $\sum_{n=1}^{\infty} \frac{1}{n}$ $\boxed{\text{diverges}}$ (can be shown that it is larger than $\int_1^{\infty} \frac{1}{x} dx = \infty$)

<u>$x = -1$</u> $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ $\boxed{\text{converges}}$ (alternate series test)

Remark: Theorem does not say anything about con/divergence for $x = \pm R$

here are examples that anything is possible for these values.

(2) $\sum_{n=0}^{\infty} x^n$ i.e. $a_n = 1 \quad \forall n.$

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = 1 = \beta = 1/R$$

$$\Rightarrow \boxed{R = 1}$$

converges for $|x| < 1$

what about $x = 1$? $\boxed{\text{diverges}}$ $(\to \infty)$

" " $x = -1$

check: partial sums

$$\sum_{n=0}^{k} (-1)^n = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

$\Rightarrow$ does $\boxed{\text{not converge}}$!

trick: set $y = x^4$

$\Rightarrow$ get series $\sum_{n=0}^{\infty} 3^{-n} y^n$

$$\Rightarrow \lim \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{3^{-n-1}}{3^{-n}} \right| = \lim 3^{-1} = \frac{1}{3}$$

$\Rightarrow$ $\beta = 1/3 \Rightarrow R = 3$

i.e. series converges if $|y| < 3$

$\Rightarrow$ originial series converges if $|x^4| < 3$

$$\Leftrightarrow \boxed{|x| < \sqrt[4]{3} = 3^{1/4}}$$

Examples:

Remark: If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists ← used for ratio test

$\Rightarrow$ it is equal to $\limsup |a_n|^{1/n}$

used for root test

usually ratio test is easier to work with!

① $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ i.e. $a_n = \frac{1}{n!}$

try ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \left(\frac{1/(n+1)!}{1/n!}\right)$

$= \lim \left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\Rightarrow \beta = 0$ and $R = \infty$

i.e. series converges for all x

similarly:

series diverges if $\beta|x| > 1 \iff |x| > 1/\beta = R$

This covers the case with $0 < \beta < \infty$.

If $\underline{\beta = 0}$: $\Rightarrow \beta|x| = 0 < 1$ for all x

$\Rightarrow$ series converges for all x

$\iff |x| < \infty = R$

If $\beta = \infty \iff R = 0 \Rightarrow \beta|x| = \infty$ (except for $x=0$)

$\Rightarrow$ series converges only for $x=0$

✓

Theorem The power series $\sum a_n x^n$

- converges for $|x| < R$
- diverges for $|x| > R$

(if $R = \infty \Rightarrow$ converges for every x
$R = 0 \Rightarrow$ only converges for $x = 0$)

Proof relies on root test for ordinary series:

it says: the series $\sum_{n=0}^{\infty} b_n$

- converges if $\limsup |b_n|^{1/n} < 1$
- diverges if " > 1

apply root test for $\boxed{b_n = a_n x^n}$

$$\Rightarrow \limsup_{n \to \infty} |b_n|^{1/n} = \limsup_{n \to \infty} |a_n x^n|^{1/n} = \limsup_{n \to \infty} |a_n|^{1/n} |x| = \beta |x|$$

$\Rightarrow$ power series converges if $\beta |x| < 1 \iff |x| < 1/\beta = R$

A few technicalities:

ⓐ $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup\{a_k,\ k \geq n\} \right)$

ⓑ $\limsup_{n\to\infty} c a_n = c \limsup a_n$ c constant number

Let $\sum a_n x^n$ be a power series

Define $\beta = \limsup_{n\to\infty} |a_n|^{1/n}$

(in this case $|a_n| \sim \text{const.}\ \beta^n$)

$\boxed{\text{radius of convergence } R}$ is defined by

$\boxed{R = 1/\beta}$

if $\boxed{\beta = 0 \Rightarrow R = \infty}$ if $\boxed{\beta = \infty \Rightarrow R = 0}$

Chapter 4: Power Series

$$\sum_{n=0}^{\infty} a_n x^n$$

a_n real numbers

x variable

<u>Central question:</u> For which values of x does the power series converge.

for $x=0$: Power series $= a_0$

We will see: three possible cases:

- only converges for $x=0$
- there exists a number $R>0$ such that
  series converges for $|x|<R$
  " diverges " $|x|>R$
- series converges for all x.